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The gravitational field of a rotating infinite cylindrical shell

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Abstract. Israel's method for treating surface layers is applied to determine the gravitational field due to a rotating cylindrical shell. The interior space–time is flat while the exterior metric can be one of three types. For a given value of the stress in the cylinder, the type of the exterior metric depends on the mass per unit coordinate length of the cylinder.

1. Introduction

The problem of determining the gravitational field due to a rotating infinite cylindrical shell has been discussed by Frehland (1972) and Papapetrou *et al* (1978). However in these papers the authors have restricted their attention to one form of the exterior metric whereas it is known (Van Stockum 1937, Tipler 1974, Bonnor 1980) that there are three real forms for the exterior metric depending on whether a certain constant of integration is positive, negative or zero. In the present work we shall use Israel's (1966) method for constructing shell sources to match, in their most general form, the three exterior forms of the metric to the interior metric which is necessarily flat (Davies and Caplan 1971). It is shown that the form of the exterior metric depends on whether the mass per unit coordinate length of the cylinder is less than, equal to or greater than a certain critical value. As a particular example we discuss briefly the case of a shell composed of dust.

In § 2 we give the three exterior vacuum metrics for a stationary cylindrically symmetric field in their most general form and show that the interior is necessarily flat. In § 3 we apply these metrics to the problem of an infinite cylindrical shell of coordinate radius $r = a$ and find the surface energy tensor and mass per unit coordinate length of the shell for each of the three exterior metrics. In § 4 we give the restrictions on the metric constants imposed by physical considerations and evaluate the proper density and principal stresses on the shell, which we then use in § 5 to show that for a given stress, the value of the mass per unit coordinate length determines the type of exterior metric.

2. General solution for a stationary cylindrically symmetric vacuum field

A stationary vacuum field with cylindrical symmetry has a metric of the form

$$(\mathrm{d}s)^2 = -\mathrm{e}^{2\lambda}(\mathrm{d}t + \nu\,\mathrm{d}\phi)^2 + \mathrm{e}^{-2\lambda}[\mathrm{e}^{2\gamma}(\mathrm{d}r^2 + \mathrm{d}z^2) + r^2\,\mathrm{d}\phi^2] \tag{2.1}$$

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where r, z, ϕ are cylindrical coordinates and λ, ν and γ are functions of r only. The vacuum field equations $R_{ij} = 0$ yield three different real forms of the metric corresponding to the three types of cylindrically symmetric solutions discussed by Van Stockum (1937), Tipler (1974) and Bonnor (1980). These are:
case (i)

$$(\mathrm{d}s)^2 = -\frac{1}{\alpha_1}[\rho^{1-c} - \rho^{1+c}]\,\mathrm{d}t^2 + \frac{2}{\alpha}\left[\left(\frac{\alpha - 2G}{2\omega}\right)\rho^{1+c} + \left(\frac{\alpha + 2G}{2\omega}\right)\rho^{1-c}\right]\mathrm{d}\phi\,\mathrm{d}t$$
$$+\frac{1}{\alpha_1}\left[\left(\frac{\alpha - 2G}{2\omega}\right)^2\rho^{1+c} - \left(\frac{\alpha + 2G}{2\omega}\right)^2\rho^{1-c}\right]\mathrm{d}\phi^2 + D\rho^{\frac{1}{2}(c^2-1)}(\mathrm{d}r^2 + \mathrm{d}z^2) \tag{2.2}$$

where $\alpha_1^2 = \alpha^2$, $\rho = |\omega| r$ and α, c, D, G and ω are constants;
case (ii)

$$(\mathrm{d}s)^2 = -\frac{2\omega r}{\alpha_1}\sin\beta\,\mathrm{d}t^2 + 2r\left(\cos\beta - \frac{2G}{\alpha}\sin\beta\right)\mathrm{d}\phi\,\mathrm{d}t$$
$$+\frac{2r\alpha}{\omega\alpha_1}\left[\left(\frac{\alpha}{4} - \frac{G^2}{\alpha}\right)\sin\beta + G\cos\beta\right]\mathrm{d}\phi^2 + D\rho^{-\frac{1}{2}(1+c^2)}(\mathrm{d}r^2 + \mathrm{d}z^2) \tag{2.3}$$

where $\rho = |\omega| r$, $\beta = c\log\rho$ and α, c, D, G and ω are constants and again $\alpha_1^2 = \alpha^2$;
case (iii)

$$(\mathrm{d}s)^2 = -2br\log\rho\,\mathrm{d}t^2 + 2r(1 - 2G\log\rho)\,\mathrm{d}\phi\,\mathrm{d}t$$
$$+(2Gr/b)(1 - G\log\rho)\,\mathrm{d}\phi^2 + D\rho^{-1/2}(\mathrm{d}r^2 + \mathrm{d}z^2) \tag{2.4}$$

where $\rho = |\omega| r$ and b, D, G and ω are constants.

In constructing a cylindrical shell source, Frehland (1972) and Papapetrou *et al* (1978) consider only case (i). In the following sections all three cases will be studied.

We notice that, by means of the complex transformation $c \to ic$, $\alpha \to i\alpha$, $\alpha_1 \to i\alpha_1$, we can obtain the case (ii) metric from case (i), as has been mentioned by Kramer *et al* (1980). Case (iii) is obtained from case (i) by letting c and α go to zero keeping α/c constant. If one lets G and b go to zero in case (iii) while keeping G/b constant one obtains cases (i) and (ii) with $c = 0$ ($\alpha \neq 0$). For the purposes of a classification to be made in §§ 4 and 5 we shall consider the latter metric to be a special limit of case (iii) so that c will be taken to be strictly non-zero in cases (i) and (ii).

Case (i) is Petrov type I for all non-zero values of c, except $c = \pm 1$ when it is flat and $c = \pm 3$ when it is Petrov type D. Case (ii) is Petrov type I for all non-zero values of c. Case (iii) is Petrov type II in general but the special limit referred to above (or alternatively cases (i) and (ii) with $c = 0$) is Petrov type D.

These three cases represent the complete general solution for a stationary vacuum field exterior to a cylindrically symmetric source. For the interior vacuum solution we require, in addition, that the curvature invariants be non-singular along the axis $r = 0$ and that elementary flatness holds along this axis, i.e. that $r^{-2}g_{\phi\phi}/g_{rr} \to 1$ as $r \to 0$ (see Synge 1964).

The only non-identically zero curvature invariants for all three cases are $R_{abcd}R^{abcd}$ and $R_{ab}{}^{cd}R_{cd}{}^{ef}R_{ef}{}^{ab}$. In case (iii) these are singular at $r = 0$ while in case (ii) they are non-singular at $r = 0$ only if $c^2 \geqslant 3$, but the metric does not satisfy elementary flatness there. Hence the interior metric cannot be of either of these two forms. In case (i) the curvature invariants are non-singular on the axis $r=0$ only if $c^2=1$, in which case the metric is Minkowskian. Thus the interior space–time must be flat.

Putting $c=\pm 1$ in (2.2) the elementary flatness condition on the axis yields two relations between the constants and by a transformation of the form $\phi'=\omega t/D+\phi$ one obtains the metric

$$ds^2=-L_0^{-1}\,dt^2+L_0r^2\,d\phi^2+L_0(dr^2+dz^2) \tag{2.5}$$

where $L_0=D$. This is the form of the interior flat metric used by Papapetrou *et al* (1978).

3. Infinite cylindrical shell

We apply Israel's (1966) method for surface layers to the interior metric (2.5) and the three exterior metrics of the previous section and thus construct, in its most general form, the gravitational field of an infinite cylindrical shell. By a straightforward argument involving the interior and exterior Killing vectors on the shell one may show that the coordinates (t, r, z, ϕ) can be taken to be the same inside and outside the shell, without loss of generality.

3.1. Case (*i*)

Let the history, Σ, of the shell be given by $r=a$. The metric on Σ induced by its embedding in the interior space–time is

$$ds_-^2=-L_0^{-1}\,dt^2+L_0a^2\,d\phi^2+L_0\,dz^2 \tag{3.1}$$

and that due to its embedding in the exterior space–time is

$$\begin{aligned}ds_+^2=&-\frac{1}{\alpha_1}(\rho_0^{1-c}-\rho_0^{1+c})\,dt^2+\frac{2}{\alpha}\left[\left(\frac{\alpha-2G}{2\omega}\right)\rho_0^{1+c}+\left(\frac{\alpha+2G}{2\omega}\right)\rho_0^{1-c}\right]d\phi\,dt\\&+\frac{1}{\alpha_1}\left[\left(\frac{\alpha-2G}{2\omega}\right)^2\rho_0^{1+c}-\left(\frac{\alpha+2G}{2\omega}\right)^2\rho_0^{1-c}\right]d\phi^2+D\rho_0^{\frac{1}{2}(c^2-1)}\,dz^2\end{aligned} \tag{3.2}$$

where $\rho_0=|\omega|a$. The condition

$$ds_-^2=ds_+^2 \tag{3.3}$$

yields three independent equations for the six unknowns L_0, α, c, D, G and ω. These are

$$L_0=D\rho_0^{\frac{1}{2}(c^2-1)} \tag{3.4}$$

$$\rho_0^{1-c}=\alpha_1(\alpha-2G)/2\alpha L_0 \tag{3.5}$$

and

$$\rho_0^{1+c}=-\alpha_1(\alpha+2G)/2\alpha L_0. \tag{3.6}$$

In general, if x_+^i $(i=0, 1, 2, 3)$ are the exterior coordinates and $x_+^i=x_+^i(\xi^\mu)$ $(\mu=0, 2, 3)$ is the equation of the shell regarded as embedded in the exterior space–time, then the second fundamental form of Σ due to this embedding is

$$K_{\mu\nu}^+=\overset{+}{n}_{i|j}\frac{\partial x_+^i}{\partial\xi^\mu}\frac{\partial x_+^j}{\partial\xi^\nu} \tag{3.7}$$

where the vertical stroke indicates covariant derivative with respect to the exterior metric and n_i^+ is a unit vector normal to Σ. In the same way the interior second fundamental form is

$$K^-_{\mu\nu} = n^-_{i|j} \frac{\partial x^i_-}{\partial \xi^\mu} \frac{\partial x^j_-}{\partial \xi^\nu} \tag{3.8}$$

where the minus signs refer in an obvious way to the interior space–time. Defining $\gamma_{\mu\nu}$ by

$$\gamma_{\mu\nu} = K^+_{\mu\nu} - K^-_{\mu\nu} \tag{3.9}$$

the surface energy tensor, $S_{\mu\nu}$, of the shell is given by

$$-\kappa S_{\mu\nu} = \gamma_{\mu\nu} - g_{\mu\nu}\gamma \tag{3.10}$$

where $g_{\mu\nu}$ is the intrinsic metric on Σ, $\gamma = \gamma^\mu{}_\mu$ and $\kappa = 8\pi$. The calculation of $S_{\mu\nu}$ is considerably simplified here since we are taking

$$(x^0_+, x^1_+, x^2_+, x^3_+) \equiv (x^0_-, x^1_-, x^2_-, x^3_-) \equiv (t, r, z, \phi) \tag{3.11}$$

and hence the intrinsic coordinates on Σ are

$$(\xi^0, \xi^2, \xi^3) = (t, z, \phi). \tag{3.12}$$

After some manipulation using (3.4), (3.5) and (3.6) we find that the non-zero components of the surface energy tensor are

$$S_{00} = \frac{1}{4\kappa a L_0^{3/2}} \left(3 + \frac{4Gc}{\alpha} - c^2\right) \tag{3.13}$$

$$S_{30} = S_{03} = -\frac{\alpha_1 c L_0^{1/2} a\omega}{\alpha^2 \kappa} = \pm \frac{a\omega c L_0^{1/2}}{\alpha\kappa} \tag{3.14}$$

and

$$S_{33} = \frac{a L_0^{1/2}}{4\kappa} \left(1 + \frac{4Gc}{\alpha} + c^2\right). \tag{3.15}$$

Adapting Whittaker's (1935) theorem to the case of a surface layer (see McCrea 1976) we define the total mass M of the shell to be

$$M = \int_\Sigma (-S^0{}_0 + S^2{}_2 + S^3{}_3)\sqrt{-g^{(3)}}\, \mathrm{d}z\, \mathrm{d}\phi. \tag{3.16}$$

Clearly the mass will be infinite, but we can calculate the mass per unit length of z, M_1, to be

$$M_1 = \mathrm{d}M/\mathrm{d}z = \tfrac{1}{4}(1 + 2Gc/\alpha). \tag{3.17}$$

This agrees with the results of Papapetrou *et al* (1978).

3.2. *Case (ii)*

In this case we take (2.5) as the interior and (2.3) as the exterior metric. Condition (3.3) yields the following three independent equations for the six unknowns L_0, α,

c, D, G and ω:

$$\sin\beta_0 = \alpha_1/2a\omega L_0 \tag{3.18}$$

$$\cos\beta_0 = \alpha_1 G/\alpha a\omega L_0 \tag{3.19}$$

and

$$D = \rho_0^{-\frac{1}{2}(1+c^2)} \tag{3.20}$$

where $\rho_0 = |\omega| a$ and $\beta_0 = c \log \rho_0$.

Continuing as in case (i) we can calculate the surface energy tensor $S_{\mu\nu}$ and find the only non-zero components to be

$$S_{00} = \frac{1}{4\kappa a L_0^{3/2}}\left(3 + \frac{4Gc}{\alpha} + c^2\right) \tag{3.21}$$

$$S_{03} = S_{30} = \frac{\alpha_1 c a\omega L_0^{1/2}}{\alpha_2 \alpha\kappa} = \pm\frac{c a\omega L_0^{1/2}}{\alpha\kappa} \tag{3.22}$$

and

$$S_{33} = \frac{a L_0^{1/2}}{4\kappa}\left(1 + \frac{4Gc}{\alpha} - c^2\right) \tag{3.23}$$

where again $\kappa = 8\pi$. The mass per unit length of z, as defined in (3.16) and (3.17), for this case is

$$M_1 = \tfrac{1}{4}(1 + 2Gc/\alpha). \tag{3.24}$$

3.3. Case (iii)

Matching the interior metric (2.5) to the exterior metric (2.4), as in the previous cases, yields three independent equations for the five unknowns L_0, b, D, G and ω. These are

$$L_0 = D\rho_0^{-1/2} \tag{3.25}$$

$$2G \log \rho_0 = 1 \tag{3.26}$$

and

$$G = abL_0 \tag{3.27}$$

where $\rho_0 = |\omega| a$. Using these, the non-zero components of the surface energy tensor $S_{\mu\nu}$ can be shown to be

$$S_{00} = \frac{1}{4\kappa a L_0^{3/2}}(3 + 4G) \tag{3.28}$$

$$S_{03} = S_{30} = abL_0^{1/2}/\kappa \tag{3.29}$$

and

$$S_{33} = \frac{aL_0^{1/2}}{4\kappa}(1 + 4G). \tag{3.30}$$

The mass per unit length of z reduces to

$$M_1 = \tfrac{1}{4}(1 + 2G). \tag{3.31}$$

4. Properties of the surface energy tensor

Following Hawking and Ellis (1973) and considering the non-zero components of the surface energy tensor, $S_{(\mu\nu)}$, with respect to the orthonormal base

$$e^{\mu}_{(0)} = L_0^{1/2}\delta^{\mu}_0 \qquad e^{\mu}_{(3)} = \frac{1}{aL_0^{1/2}}\delta^{\mu}_3 \tag{4.1}$$

we find that the restrictions imposed on $S_{(\mu\nu)}$ by the energy conditions are

$$\begin{aligned} &S_{(00)} \geqslant 0 \qquad S_{(00)} \geqslant |S_{(33)}| \\ &S_{(00)} + S_{(33)} \geqslant 2|S_{(03)}| \qquad \text{and} \qquad S_{(00)} \geqslant |S_{(03)}|. \end{aligned} \tag{4.2}$$

Applying these to the surface energy tensors in each of the three cases one obtains the following results.
Case (i)

$$c^2 \leqslant 1 \tag{4.3}$$

and

$$1 + 4Gc/\alpha + c^2 \geqslant 0. \tag{4.4}$$

Case (ii)

$$1 + 4Gc/\alpha - c^2 \geqslant 0. \tag{4.5}$$

Case (iii)

$$1 + 4G \geqslant 0. \tag{4.6}$$

Clearly these restrictions ensure that the mass per unit coordinate length is positive for each of the three cases, as given by (3.17), (3.24) and (3.31).

The proper surface density μ and the principal stresses in the z and ϕ directions, written σ_z and σ_ϕ respectively, can now be obtained from the eigenvalues of the surface energy tensor. It is convenient to use the orthonormal components $S_{(\mu\nu)}$ defined above. Since $S_{(2\mu)} = 0$, $\sigma_z = 0$ in all three cases and the eigenvector equation reduces to the simple 2×2 tensor equation

$$S_{(AB)}u^B = \lambda \eta_{AB} u^B \tag{4.7}$$

where A, $B = 0$, 3 and $\eta_{AB} = \mathrm{diag}(-1, 1)$. Solving (4.7) yields two eigenvectors, one time-like and one space-like with the corresponding eigenvalues $\lambda_{(0)}$ and $\lambda_{(3)}$ respectively. The proper surface density $\mu = -\lambda_{(0)}$ and the principal stress in the ϕ direction $\sigma_\phi = -\lambda_{(3)}$. It is found that for all three cases

$$\mu = p(q + 2\sqrt{8M_1 - q}) \tag{4.8}$$

$$\sigma_\phi = p(q - 2\sqrt{8M_1 - q}) \tag{4.9}$$

where

$$p = 1/4\kappa a L_0^{1/2} \qquad \text{and} \tag{4.10}$$

$$q = 1 - c^2 \qquad 1 \geqslant c^2 > 0 \qquad \text{for case (i)} \tag{4.11}$$

$$q = 1 + c^2 \qquad c^2 > 0 \qquad \text{for case (ii)} \tag{4.12}$$

$$q = 1 \qquad \text{for case (iii)}. \tag{4.13}$$

M_1 is given by (3.17), (3.24) and (3.31) for cases (i), (ii) and (iii) respectively.

A simple calculation in each of the three cases shows that the limitations on the constants contained in the previous section ensure that μ is real and positive and that σ_ϕ is real in each case, so the densities and stresses are physically reasonable.

5. General discussion

In cases (i) and (ii) matching the interior and exterior metrics for a given radius yields three equations for six unknowns, so we require three further conditions to determine the metric completely. This is reasonable since the physical quantities such as mass and stress will affect the metric. If we fix the mass per unit length M_1, the density μ and the stress σ_ϕ we can determine all the constants and so both interior and exterior metrics are known.

In case (iii) however we have only five unknowns and so for a given radius, if two of the physical quantities are fixed we can evaluate all the constants and hence the metric.

A further interesting point is that given σ_ϕ/p, the value of the mass per unit length M_1 determines whether the exterior metric is case (i), (ii) or (iii). We can show this by writing M_1 in terms of σ_ϕ using (4.9) which results in the equation

$$M_1 = \tfrac{1}{32}[(q-z)^2 + 4q] \tag{5.1}$$

where $z = \sigma_\phi/p$, and $0 \leq q < 1$ in case (i), $q > 1$ in case (ii) and $q = 1$ in case (iii). Since, by (4.9), $z \leq q$ in all three cases, we obtain the following general classification.

For $z \leq 0$

In case (i) $$\tfrac{1}{32}z^2 \leq M_1 < \tfrac{1}{32}(z^2 - 2z + 5) \tag{5.2}$$

in case (ii) $$M_1 > \tfrac{1}{32}(z^2 - 2z + 5) \tag{5.3}$$

in case (iii) $$M_1 = \tfrac{1}{32}(z^2 - 2z + 5). \tag{5.4}$$

For $0 < z < 1$

In case (i) $$\tfrac{1}{8}z \leq M_1 < \tfrac{1}{32}(z^2 - 2z + 5) \tag{5.5}$$

in case (ii) $$M_1 > \tfrac{1}{32}(z^2 - 2z + 5) \tag{5.6}$$

in case (iii) $$M_1 = \tfrac{1}{32}(z^2 - 2z + 5). \tag{5.7}$$

For $z = 1$

Case (i) is not possible since $z \leq q < 1$,

in case (ii) $$M_1 > \tfrac{1}{8} \tag{5.8}$$

in case (iii) $$M_1 = \tfrac{1}{8}. \tag{5.9}$$

For $z > 1$

Only case (ii) can occur and $M_1 > \tfrac{1}{8}z$. $$\tag{5.10}$$

We can see that provided z is fixed, then the value of M_1 determines whether the exterior metric is case (i), (ii) or (iii). In the limit of case (iii) where G and b go to zero while keeping G/b constant, $z = -1$ and $M_1 = \tfrac{1}{4}$.

For the purpose of comparison, consider a cylindrical shell composed of dust, as discussed by Papapetrou *et al* (1978). The stress σ_ϕ will, by definition of a dust, be zero (this is equivalent to the condition in the above paper that $T_{00}T_{33} = T_{03}{}^2$) and the inequalities (5.2), (5.3), (5.4) reduce to

$$0 \leq M_1 < \tfrac{5}{32} \qquad \text{for case (i)}$$

$$M_1 > \tfrac{5}{32} \qquad \text{for case (ii)}$$

and

$$M_1 = \tfrac{5}{32} \qquad \text{for case (iii).}$$

The extension of the solution to three exterior metrics completes the picture and allows a full range of values for the mass per unit length M_1, rather than the restricted range of case (i) as studied by Papapetrou *et al* (1978).

We note finally that if $u^a \equiv (u^0, 0, 0, u^3)$ are the orthonormal tetrad components of the time-like eigenvector (i.e. the four-velocity) then

$$(u^3/u^0)^2 = (4M_1 - \sqrt{8M_1 - q})/(4M_1 + \sqrt{8M_1 - q}) \tag{5.11}$$

so that $(u^3/u^0)^2 \to 1$ as $q \to 8M_1$. From (4.8) and (4.9) it follows that in all three cases the four-velocity becomes null as $\sigma_\phi \to \mu$.

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